The Parallel Postulate

Problems from the History of Mathematics

Lecture 3 — January 31, 2018

Brown University

Euclid's Elements

Euclid's Elements

Euclid's *Elements* is a collection of thirteen books in geometry compiled by the mathematician Euclid c. 300 BC. Roughly, these books treat:

I	General Plane Geometry	VIII	Geometric Progressions
II	'Geometric Algebra'	IX	Primes/Perfect Numbers
III	Properties of Circles	X	Irrationality
IV	Regular Polygons	XI	3D Figures/Parellepipeds
V	Proportion	XII	Simple Volumes
VI	Similar Figures	XIII	Platonic Solids
VII	Number Theory		

Euclid's Elements

The actual content of *Elements* was largely known prior to Euclid.[1]

As to the matter of the work. The geometrical part is to a large extent a compilation from the works of previous writers. Thus the substance of books I and II (except perhaps the treatment of parallels) is probably due to Pythagoras; that of book III to Hippocrates; that of book V to Eudoxus; and the bulk of books IV, VI, XI, and XII to the later Pythagorean or Athenian schools. But this material was rearranged, obvious deductions were omitted (for instance, the proposition that the perpendiculars from the angular points of a triangle on the opposite sides meet in a point was cut out), and in some cases new proofs substituted. Book X, which deals with irrational magnitudes, may be founded on the lost book of Theaetetus; but probably much of it is original, for Proclus says that while Euclid arranged the propositions of Eudoxus he completed many of those of Theaetetus. The whole was presented as a complete and consistent body of theorems.

[1] W. W. Rouse Ball, *A Short Account of the History of Mathematics*.

Euclid's Elements

The legacy of *Elements* stems from its

a. encyclopedic treatment of classical Greek geometry
b. consistent proof structure (enunciation, statement, construction, proof, conclusion)
c. consistent standard of rigor
d. use of clearly presented axioms, common notions, and definitions

Elements was used as a geometry textbook until the 19th century. This is possibly because it was written for this exact purpose.

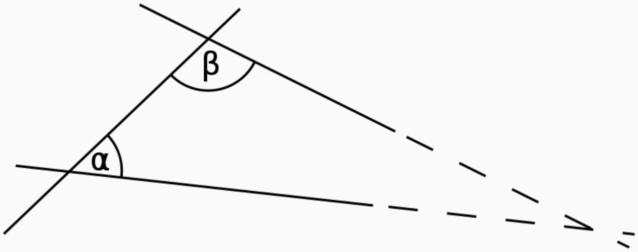
The Axioms of Elements

Euclid's *Elements* derives theorems (called 'Propositions') from codified rules of logic (called 'common notions') and axioms (called 'Postulates').

The five postulates of *Elements* are:

1. To draw a straight line from any point to any point.
2. To produce a finite straight line continuously in a straight line.
3. To describe a circle with any center and distance.
4. That all right angles are equal to each other.
5. That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

The Parallel Postulate in Elements



Euclid's fifth postulate is more commonly known as the parallel postulate.

It must have given Euclid some pause, as it is not used in the first 28 Propositions of Book I. (Today, plane geometry that uses only axioms i-iv is known as absolute geometry.)

Attempts to Prove the Parallel Postulate

Equivalent Forms of the Parallel Postulate

Euclid does not comment on the parallel postulate outright, but does prove many related facts, such as its converse (Book I, Proposition 17).

After Euclid, many Greek mathematicians (Ptolemy, Proclus, etc.) sought to prove the parallel postulate using the first four axioms of Euclid.

These attempts failed, usually by assuming an 'obvious' fact which relied on the parallel postulate.[2] In a few cases, these assumptions were provably *equivalent* to the parallel postulate.

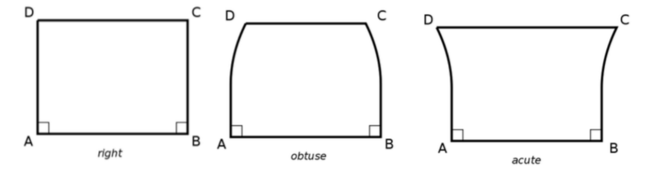
[2] It does not help that there are many intuitive definitions of parallelism and not all are equivalent via axioms i-iv.

Equivalent Forms of the Parallel Postulate

1. There is at most one line that can be drawn parallel to another given one through an external point. (Playfair's Axiom)
2. The sum of the angles in every triangle is π. (Triangle Postulate)
3. There exists a triangle whose angles add up to π.
4. The sum of the angles is the same for every triangle.
5. There exists a pair of similar, but not congruent, triangles.
6. Every triangle can be circumscribed.
7. If three angles of a quadrilateral are right angles, then the fourth angle is also a right angle.
8. There exists a quadrilateral in which all angles are right angles, that is, a rectangle.
9. There exists a pair of straight lines that are at constant distance from each other.
10. Two lines that are parallel to the same line are also parallel to each other.
11. In a right-angled triangle, the square of the hypotenuse equals the sum of the squares of the other two sides (Pythagorean Theorem).
12. There is no upper limit to the area of a triangle. (Wallis' Axiom)
13. If a line intersects one of two parallel lines, both of which are coplanar with the original line, then it also intersects the other. (Proclus' Axiom)

Khayyam–Saccheri Quadrilaterals

In *Commentary on the Difficulties of Certain Postulates in Euclid's Work*, Omar Khayyam (1048-1131) introduced what we now call Saccheri quadrilaterals.



Khayyam proved that $\angle ADC = \angle BCD$. One may then classify these quadrilaterals into three cases: acute, right, and obtuse.

The obtuse case is eventually ruled out by Proposition 17 of Book I: *the sum of two angles in a triangle is less than two right angles*.

Hyperbolic Geometry

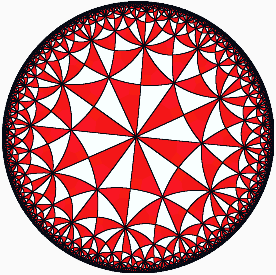
Three mathematicians have claims to inventing hyperbolic geometry:

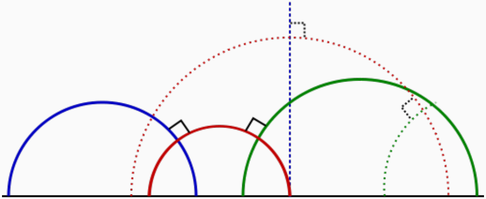
1. Johann Bolyai, *The Science of Absolute Space*, 1829.
2. Nicolai Lobachevsky, *On the Principles of Geometry*, 1829–1830.
3. Carl Friedrich Gauss, in a letter to Bolyai's father:

> If I begin with the statement that I dare not praise such a work, you will of course be startled for a moment: but I cannot do otherwise; to praise it would amount to praising myself; for the entire content of the work, the path which your son has taken, the results to which he is led, coincide almost exactly with my own meditations which have occupied my mind for from thirty to thirty-five years. On this account I find myself surprised to the extreme.

Hyperbolic Geometry

Hyperbolic geometry represents the acute case in Saccheri's treatment of quadrilaterals. Here, the angles in a triangle sum to less than π, parallel lines diverge, and triangles have bounded area.

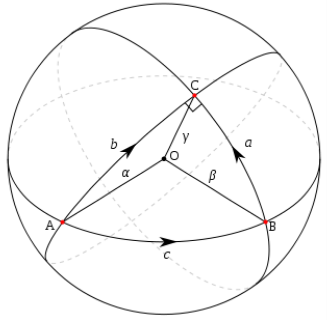




Elliptic Geometry

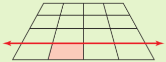
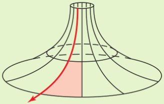
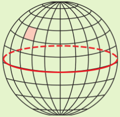
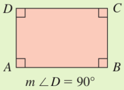
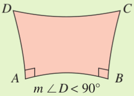
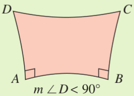
Elliptic geometry is a second non-Euclidean geometry which replaces the parallel postulate with the notion that no two lines are parallel.

In one model of elliptic geometry (often called spherical geometry), points are replaced by pairs of antipodal points on a sphere and lines are great circles on the sphere.



Elliptic geometry appears to contradict Saccheri's dismissal of the 'obtuse' case. This isn't a real contradiction, as elliptic geometry also violates Euclid's second postulate.

Comparison of Geometries

Comparison of Major Two-Dimensional Geometries		
Euclidean Geometry	**Hyperbolic Geometry**	**Elliptic Geometry**
Euclid (about 300 B.C.)	Gauss, Bolyai, Lobachevsky (ca. 1830)	Riemann (ca. 1850)
Given a point not on a line, there is one and only one line through the point parallel to the given line.	Given a point not on a line, there are an inifinite number of lines through the point that do not intersect the given line.	There are no parallels.
A representative line in each geometry is shown in color for each model, and the shaded portion showing a Saccheri quadrilateral is shown directly below the representative models.		
Geometry is on a plane:	**Geometry is on a pseudosphere:**	**Geometry is on a sphere:**
		
Lines are infinitely long.	Lines are infinitely long.	Lines are finite length.
		
The sum of the angles of a triangle is 180°.	The sum of the angles of a triangle is less than 180°.	The sum of the angles of a triangle is more than 180°.

[3] Smith, *The Nature of Mathematics, 12th Edition*, p. 380.

Aftermath

Elliptic geometry is credited to Riemann and Cayley (c. 1860). Riemann went one step further and defined what we now know of as manifolds.

The independence of the parallel postulate (from the first four axioms) was finally shown in 1868, by Beltrami. (In particular, this showed that hyperbolic geometry was consistent.)

Euclidean geometry was brought up to the standards of modern proof by David Hilbert in 1899. He laid out 20 postulates which could be used to fix gaps in Euclid's reasoning (about continuity and intersection, mostly).

Questions?