Classification of Regular Polyhedra

Problems from the History of Mathematics
Lecture 5 — February 7, 2018

Brown University
Platonic Solids
Polyhedra\(^1\), surfaces with polygonal faces, are the generalization of polygons to three-dimensional space. Adapting our definition from the planar case, we say that a polyhedron is regular if either of the following (equivalent) conditions holds:

1. all faces are congruent regular polygons and each vertex borders the same number of faces
2. its rotational symmetry group acts transitively on the sets of vertices, edges, and faces

Strange things can happen if the solid is not convex:

A Platonic solid is a convex, regular polyhedron.

\(^1\)Greek: *many bases*
\(^2\)Image Credit: *Stella.*
Discovery of the Platonic Solids

The Platonic solids have been described since antiquity. Proclus credits their discovery to Pythagoras. Others suggest that Pythagoras knew only of the tetrahedron, cube, and dodecahedron and that the octahedron and icosehedron were first due to Theaetetus (c. 417-368 BC).³

In any case, the first proof of the classification of the Platonic solids is due to Theaetetus. It is reproduced in Book XIII of *Elements*.

³Theaetetus is also known for writing the source material for much of Book X of *Elements*, which concerns irrationality.
Euclid’s classification of the five Platonic solids\(^4\) runs as follows:

‘No other figure, besides the said five figures, can be constructed which is contained by equilateral and equiangular figures equal to one another.’

1. If the regular polygon used is a triangle, we must use 3, 4, or 5 triangles at each vertex \(\Rightarrow\) (tetrahedron, octahedron, icosahedron)

2. If the regular polygon used is a square, we must use 3 at each vertex \(\Rightarrow\) (cube)

3. If the regular polygon used is a pentagon, we must use 3 at each vertex \(\Rightarrow\) (dodecahedron)

4. Other regular polygons are excluded by considering vertex angles

Existence/construction of these polyhedra is given in Propositions 13-17.

\(^4\)Proposition 18, Book XIII
Classification via Euler Characteristic

A second proof of the classification of Platonic solids uses the Euler characteristic of a polyhedron,

\[ \chi := V - E + F, \]

in which \( V, E, \) and \( F \) are the numbers of vertices, edges, and faces of the surface, respectively.

Euler's Polyhedron Formula (1750) gives \( \chi = 2 \) for convex polyhedra. Regularity implies that

1. \( nV = 2E, \) where \( n > 2 \) is the number of edges about each vertex.
2. \( mF = 2E, \) where \( m > 2 \) is the number of edges about each face.

Thus

\[ 2 = \frac{2E}{n} - E + \frac{2E}{m} \implies \frac{1}{2} + \frac{1}{E} = \frac{1}{n} + \frac{1}{m} \implies \frac{1}{n} + \frac{1}{m} > \frac{1}{2}, \]

which has solutions \((n, m) = (3, 3), (3, 4), (3, 5), (4, 3), (5, 3)\).
Quasiregular/Archimedean Solids
The list of Platonic solids may be extended by assuming only that each face is a regular polygon and that each vertex configuration be identical.\textsuperscript{5}

The classification of these 13 quasi-regular solids was first done by Archimedes (c. 287-212 BC), after whom they are now known. His work does not survive but is mentioned in Pappus’ *Collection* (c. 340 AD).

*Collection* was forgotten in the decline of European mathematics until 1588, when it was translated from Greek/Arabic to Latin by Federico Commandino (and then ignored).

Yet interest in Archimedean solids grew during the Renaissance, which led to their independent rediscovery by Kepler in 1619.

\textsuperscript{5}Ie. that there exist a rotational symmetry that acts transitively on the vertices.
Rediscovery of the Archimedean Solids

The Archimedean polyhedra in Renaissance sources. The first column shows the number given to the solid in the diagrams in J. Kepler, *Harmonice mundi* (Linz, 1619), Book 2, Proposition 28 (reproduced in our Figures 1 and 2). The second column gives the modern name of the solid. The last four columns indicate whether the solid appears in the work of Piero della Francesca (T if in his *Trattato*, L if in *Libellus*), Luca Pacioli (*De divina proportione*, 1509), Albrecht Dürer (*Underweysung*, 1525), Daniele Barbaro (*La Pratica della perspettiva*, 1568, 1569).

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Rediscovery of the Archimedean Solids

The 13 Archimedean solids are depicted below:
Creating Archimedean Solids

1. **Truncating** Platonic Solids

2. **Cantellating** Platonic Solids

3. **Rectifying** Platonic Solids
Creating Archimedean Solids

4. Truncating (Previously Rectified) Archimedean Solids

5. Snubbing (Cantellating and Deformation Twisting) Platonic Solids
Questions?