# Greek Proto-Calculus

Problems from the History of Mathematics

Lecture 6 — February 9, 2018

Brown University

# The Contributions of Eudoxus

# Infinity in Greek Mathematics

According to tradition, the Greeks were afraid of working with infinite processes by the paradoxes of Zeno (c. 450 BC). These include

1. Achilles and the Tortoise "In a race, the quickest runner can never overtake the slowest, since the pursuer must first reach the point whence the pursued started, so that the slower must always hold a lead." – as recounted by Aristotle, Physics VI:9, 239b15
2. The Dichotomy Paradox "That which is in locomotion must arrive at the half-way stage before it arrives at the goal." – as recounted by Aristotle, Physics VI:9, 239b10
3. The Arrow Paradox "If everything when it occupies an equal space is at rest, and if that which is in locomotion is always occupying such a space at any moment, the flying arrow is therefore motionless." – as recounted by Aristotle, Physics VI:9, 239b5

# Eudoxus' Theory of Proportions

Eudoxus' key observation is that lengths are determined by the **rational** lengths less than them and those greater than them. For instance, $\sqrt{2}$ is determined by the two sets

$$L_{\sqrt{2}} = \{x \in \mathbb{Q} : x^2 < 2\}, \quad U_{\sqrt{2}} = \{x \in \mathbb{Q} : x^2 > 2\}.$$

Dedekind formalized this idea in 1872, when he gave a model for the real numbers as the set of pairs $\{L, U\}$ for which

a. $L$ and $U$ partition $\mathbb{Q}$
b. $\ell < u$ for all $\ell \in L$ and $u \in U$.

This construction is now known as a Dedekind cut.

# The Method of Exhaustion

Eudoxus' Theory of Proportions is best illustrated by its analogue in two dimensions, where the area of planar regions are determined by collections of inscribed and circumscribed figures.[1] This generalization is known as the Method of Exhaustion and is also credited to Eudoxus.

Exhaustion is the main tool used in Book XII of *Elements*. For example,

**Proposition 1:**

Similar polygons inscribed in circles are to one another as the squares on the diameters.

**Proposition 2:**

Circles are to one another as the squares on the diameters.

Proposition 2 is proven by Exhaustion: we approximate the area of the circle by polygons and apply Proposition 1.

[1] These figures are usually polygons or, in three dimensions, polyhedra.

# The Method of Exhaustion in Elements

Exhaustion is also used in Book XII to prove several results about volume:

- Prop. 5: The volumes of two tetrahedra of the same height are proportional to the areas of their triangular bases.

- Prop. 10: The volume of a cone is a third of the volume of the corresponding cylinder which has the same base and height.

- Prop. 11: The volume of a cone (or cylinder) of the same height is proportional to the area of the base.

- Prop. 12: The volume of a cone (or cylinder) that is similar to another is proportional to the cube of the ratio of the diameters of the bases.

- Prop. 18: The volume of a sphere is proportional to the cube of its diameter.

Note that these results **do not** give explicit constants of proportionality.

# The Method of Mechanical Theorems

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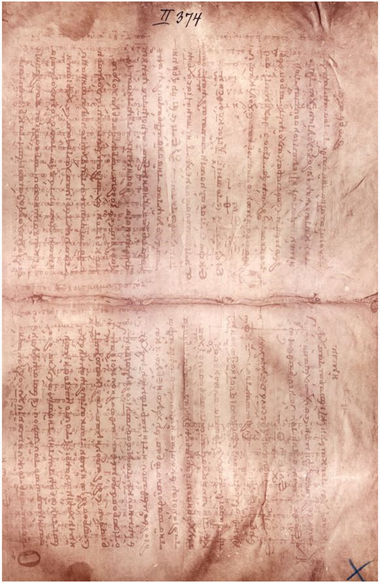
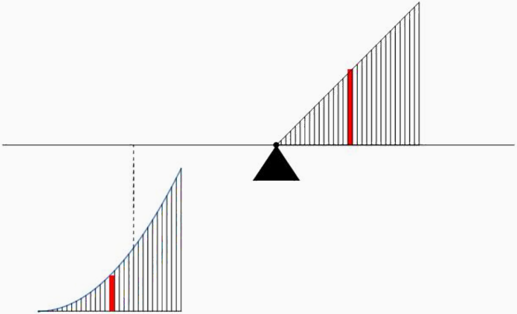
*The Method* is one of the major surviving works of Archimedes (c. 287 - 212 BC). Originally thought lost, it was recovered beneath a liturgical text (now called the Archimedes Palimpsest) in 1906.

The name derives from Archimedes' use of mechanical principles (such as the law of the lever) to intuit mathematical results. His proofs are then shown to be valid using exhaustion.



# Law of the Lever

A typical application of the law of the lever can be seen in Archimedes' calculation of the area under a parabola.



The lever balances because the infinitesimal contribution of weight $x$ at $x$ balances with weight $x^2$ at $-1$. It remains to calculate the torque from the triangle, which is its total mass $(1/2)$ times the distance from fulcrum to the center of gravity $(2/3)$. Thus the parabola has weight $1/3$.

**Questions?**