PROBLEMS FROM THE HISTORY OF MATHEMATICS
PROBLEM SET #2

DUE FRIDAY, 2/9/2018

Exercise 1. In this exercise we will prove that Euclid's fifth postulate implies Playfair's axiom. Recall that Euclid's axiom states:

> *If a line segment intersects two straight lines forming two interior angles on the same side that sum to less than two right angles, then the two lines, if extended indefinitely, meet on that side on which the angles sum to less than two right angles.*

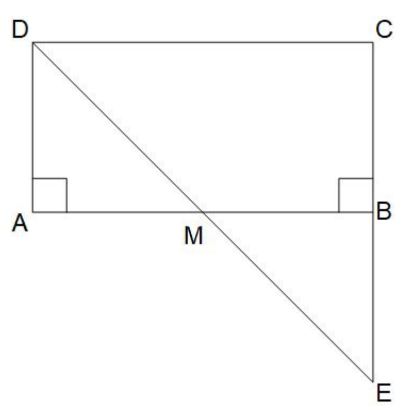
Playfair's axiom is the following:

> *In a plane, given a line and a point not on it, at most one line parallel to the given line can be drawn through the point.*

Complete the following:

1. Given a line L and a point P not on L, construct a perpendicular to L through P called M and a perpendicular to M through P called N. Why must L be parallel to N? *Hint: Apply Proposition 16 from Book I.*
2. Now suppose that N' is a different line through P. Use the parallel postulate to prove that N' intersects L, a contradiction. (So N was unique and the parallel postulate implies Playfair's axiom.)

Exercise 2. In this exercise we will prove that Saccheri quadrilaterals cannot have obtuse angles. Consider the Saccheri quadrilateral $ABCD$ below, where $AD = BC$. We extend BC to E such that $BC = EB$. Let M be the midpoint of AB and draw EM and MD.



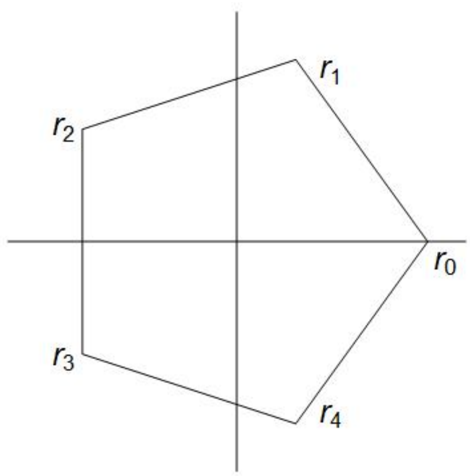
Prove the following:

1. Prove that $\triangle ADM$ is similar to $\triangle BEM$.
2. Prove that M lies on DE. *Hint: if not, extend DM to E' and compare $\angle BME$ to $\angle BME'$. Apply Proposition 15 of Book I.*
3. The sum of the summit angles in $ABCD$ is $m\angle ADC + m\angle BCD$. Prove that this equals the sum of the angles in $\triangle EDC$.
4. Conclude that the summit angles cannot both be obtuse. *Hint: Look up the Saccheri–Legendre theorem, understand its proof, and apply the result.*

Exercise 3. In spherical geometry, the angles in a triangle need not sum to π. Describe the range of possible angle sums.

Exercise 4. In this exercise, we adapt Gauss' construction of the heptadecagon to construct a pentagon.

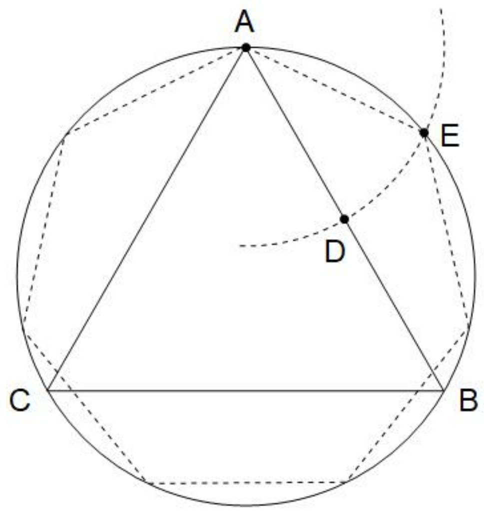
1. Associate each vertex r_j of the regular pentagon to a point on the unit circle in the complex plane. If one vertex lies at 1, prove that the others lie at roots of $z^4 + z^3 + z^2 + z + 1$.



2. Show that $r_1 + r_4$ and $r_2 + r_3$ are roots of the polynomial $y^2 + y - 1$. Use this to find an algebraic expression for $r_1 + r_4$ in radicals. How does your answer relate to $\cos(2\pi/5)$?
3. Give an algorithm for producing a regular pentagon. (You don't have to construct one but a sketch may help describe what's happening.)

Exercise 5. This exercise studies an approximate construction for a regular heptagon which appears in *Four Books on Measurement*, a treatise written by Dutch artist Albrecht Dürer which was published in 1525.

Construct an equilateral triangle ABC inscribed in a unit circle. Bisect AB at D and draw the circle with center A and radius AD. This circle intersects the unit circle at E.



The points A and E represent two adjacent vertices of our 'regular heptagon.' Drawing a circle at E of radius EA can give a third vertex, etc. If this process is continued until we loop back near A, our final vertex is almost exactly at A. How close is the final vertex to A? (Approximate.)