PROBLEMS FROM THE HISTORY OF MATHEMATICS
PROBLEM SET #5

DUE FRIDAY, 3/2/2018

Exercise 1. Let $f(x) = x^3 + px + q$. In this exercise, we derive the cubic formula in the form known to Cardano.

a. Substitute $x = u + v$ into $f(x)$ and show that $f(x) = 0$ implies
$$u^3 + v^3 + (3uv + p)(u + v) + q = 0.$$
b. Assume as well that u, v satisfy $3uv + p = 0$. As in class, combine these equations to relate u^3 and v^3 in a quadratic equation.
c. Solve this equation to derive Cardano's cubic formula.

Exercise 2. Let $f(x) = x^3 - 15x - 4$. (This is Bombelli's example.)

a. Use the cubic formula (in the form we gave in class) to find a closed form for the real root $x_0 \approx -0.2654$ of $f(x)$.
b. Find a closed form expression for the root you obtain by applying Cardano's formula (1c) to $f(x)$. Do you obtain the same root?

Exercise 3. Assume that every non-constant polynomial in $\mathbb{R}[z]$ has a root. Prove that every non-constant polynomial in $\mathbb{C}[z]$ has a root.

Exercise 4. Use Euler's factorization of real quartics to find closed form solutions for the four (complex) roots of $z^4 + 4z^2 - 4z + 1$. *Hint: the cubic you find along the way has one particularly nice root.*

Exercise 5. The Fundamental Theorem of Algebra may be obtained as a corollary to *Liouville's Theorem*, which states that the only bounded holomorphic functions $f : \mathbb{C} \to \mathbb{C}$ are constants. Show this implication. (You may assume that rational functions are holomorphic away from poles.)

Exercise 6. Let $a(z)$ and $b(z)$ be polynomials in $\mathbb{C}[z]$. If $b \neq 0$, prove that there exist polynomials $q(z)$ and $r(z)$ such that

i. $a(z) = q(z)b(z) + r(z)$
ii. $\deg r < \deg b$.