The Basel Problem

Problems from the History of Mathematics
Lecture 17 — March 21, 2018

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The first infinite series to be considered hail from ancient Greece. We have already mentioned two of these series:

a. The geometric series in Zeno’s Achilles paradox:

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1.$$

b. The geometric series of Archimedes’ quadrature of the parabola:

$$\sum_{n=0}^{\infty} \frac{1}{4^n} = 1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \cdots = \frac{4}{3}.$$

Despite these early results, the geometric series was not understand in general until the work of Mercator and Newton in the 1600s.
Early Results in Infinite Series

The next series we consider is due, independently, to Nicole Oresme and Richard Suiseth. Around 1350, each showed that

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \cdots = 2.$$  

Oresme’s argument is geometric and particularly simple.

By summing horizontal strips, we obtain a geometric series. Summing vertical strips gives desired result.
Divergence of the Harmonic Series

In the same work, Oresme gives a simple proof that the harmonic series diverges. This is because

\[
1 + \frac{1}{2} + \left( \frac{1}{3} + \frac{1}{4} \right) + \left( \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \right) + \cdots
\]

\[
> 1 + \frac{1}{2} + \left( \frac{1}{4} + \frac{1}{4} \right) + \left( \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} \right) + \cdots
\]

The second, smaller series diverges, since each parenthetical sums to \( \frac{1}{2} \).

**Note:** Oresme’s result shows that

\[
\sum_{n \leq X} \frac{1}{n} \gtrsim \frac{1}{2} \log_2 n.
\]

The series is asymptotic to \( \log n \), so Oresme’s bound is tight up to the leading constant. Oresme’s work on the harmonic series is the first series work on the divergence of infinite series.
Infinite series became far more common after the advent of power series, which generalized many of the series known up to that point.

Power series begin with Madhava’s work in India but really took off only after their discovery in the West by Mercator and Newton.\(^1\)

Here, the key series was Newton’s Binomial Series:

\[
(1 + x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k, \quad \binom{\alpha}{k} = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)}{k!}.
\]

This series predates Taylor’s theorem and was first proven by studying the forward difference operator \(\Delta_h f(x) = f(x + h) - f(x)\), a sort of discrete derivative.

\(^1\)This is another subject to benefit from the pursuit of closed forms for integrals.
Finding Sums
With the exception of Archimedes’ quadrature of the parabola, these early results are exclusively decomposition problems: one begins with a known number/function and then recognizes it as an infinite sum.

This was helpful to the development of theory because it avoids some of the issues of divergent series. ²

Questions about the closed forms of series remained relatively unexplored between the time of Archimedes and 1650, when Mengoli published the book *Novae Quadraturae Arithmeticae*. Here, Mengoli introduced the telescoping series

\[
\sum_{n=1}^{\infty} \frac{1}{n(n + 1)} = \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n + 1} \right) = 1.
\]

²Which we will talk about next time!
The Basel Problem

In the same work, Mengoli studied the harmonic series and its relative, the harmonic series of squares:

\[ \zeta(2) := \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots \]

Mengoli attempts to find a closed form for the sum of this series but fails. This problem appears (likely independently) in the work of Wallis in 1655 and quickly gained notoriety across Europe, perhaps due to the superficial resemblance to the telescoping series of Mengoli.

We know this problem now as the Basel Problem:\(^3\)

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\[^3\]The name comes from Basel, Switzerland, residence of the Bernoullis and birthplace of Leonhard Euler.
In 1728, Daniel Bernoulli wrote of a way to approximate $\zeta(2)$ and showed that its value was approximately 1.6. In response, Goldbach proved that

$$1.64 < \zeta(2) < 1.66.$$  

It’s probable that Euler first heard of the problem from Daniel Bernoulli. In any case, his first contribution to the problem is the development of a faster series which converges to $\zeta(2)$ (c. 1731).

Briefly, Euler begins with the series

$$\sum_{n=1}^{\infty} \frac{x^n}{n^2} = - \int_0^1 \frac{\log(1 - x)}{x} \, dx.$$  

By splitting the integral at $\alpha$ and integrating by parts, Euler proved that

$$\zeta(2) = \log \alpha \log(1 - \alpha) + \sum_{n=1}^{\infty} \frac{\alpha^n}{n^2} + \sum_{n=1}^{\infty} \frac{(1 - \alpha)^n}{n^2}.$$
Euler’s Closed Form

Euler’s first breakthrough in the Basel Problem came in 1734 and made him an overnight celebrity. The key insight was to extend the work of Newton, Lagrange, and others in elementary symmetric polynomials to functions with infinitely many roots.

We consider the function \( f(x) = (\sin \sqrt{x})/\sqrt{x} \). From Newton, we have

\[
f(x) = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots
\]

But \( f(x) \) may also be considered as a product of linear factors:

\[
f(x) = \prod_{n=1}^{\infty} \left(1 - \frac{x}{\pi n}\right) \left(1 + \frac{x}{\pi n}\right) = \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{\pi^2 n^2}\right).
\]

“Equating coefficients” gives

\[
-\frac{1}{6} = -\frac{1}{\pi^2} - \frac{1}{4\pi^2} - \frac{1}{9\pi^2} - \cdots = -\frac{1}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2}.
\]
Later proofs of Euler’s result appeared which put it on solid analytic foundation. These later proofs are quite diverse and build upon:

a. de Moivre’s theorem to relate the partial sums of $\sum 1/n^2$ to identities for cotangent and cosecant (due to Cauchy)
b. Integral identities for $(\arcsin x)^2$
c. The Fourier transform (and in particular, Parseval’s Theorem)
d. The functional equation of the Riemann zeta function:

$$\zeta(1 - s) = \pi^{-s}2^{1-s}\Gamma(s)\cos\left(\frac{\pi s}{2}\right)\zeta(s),$$

conjectured by Euler in 1749 and proven by Riemann in 1859.
Euler’s proof extends to give closed forms for the values

\[ \zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4}, \quad \zeta(6) = \sum_{n=1}^{\infty} \frac{1}{n^6}, \quad \ldots \]

but does not address the odd cases. Little is known about the odd case even today, because ideas that work in the even case require evenness to create symmetry.

Roger Apéry proved that \( \zeta(3) \) was irrational in 1978 (a result known as Apéry’s theorem), but even the irrationality of \( \zeta(5) \) remains open.
Questions?