PROBLEMS FROM THE HISTORY OF MATHEMATICS
PROBLEM SET #6

DUE FRIDAY, 3/9/2018

Exercise 1. The following theorem appears in the work of Galois:

Theorem. *Let $f(x) \in k[x]$ be an irreducible polynomial of prime degree. Then $f(x) = 0$ is solvable in radicals if and only if all the roots of P can be rationally expressed over k from any two of them.*

a. Fix $f(x) \in \mathbb{Q}[x]$ of prime degree and suppose that $f(x)$ is solvable in radicals. Using the theorem above, what are the possible numbers of real roots of $f(x)$?
b. Use your results from (a) to produce a polynomial $f(x) \in \mathbb{Q}[x]$ which is not solvable in radicals. Make sure you check that your polynomial is irreducible.

Exercise 2. Using theorem above, prove that any cubic equation must be solvable by radicals.

Exercise 3. Our proof of the Abel–Ruffini Theorem concluded with the claim that the splitting field $L = k(\alpha_1, \ldots, \alpha_n)$ of a polynomial $f(x) \in k[x]$ with Galois group S_n had no radical tower for $n \geq 5$. In the supposed tower

$$k = K_0 \subset K_1 \subset \cdots \subset K_m = L,$$

we showed that K_1/K_0 was a quadratic extension and that K_1 had no radical extension in L. This last bit used the following claim:

Claim. *If $K_2 = K_1(\beta)$ with $\beta^q = b$ and β is fixed by all of A_n, then $\beta \in K_1$.*

This claim is obvious if we assume some Galois theory, as K_1 is the fixed field of A_n. In this exercise, we see how to prove this claim without invoking the Galois correspondence.

a. Suppose that $\beta \in K_1$ is fixed by all of A_n and let $\tau \in S_n$ be a transposition. Prove that $\beta + \tau(\beta)$ and $\beta\tau(\beta)$ are fixed by each element of S_n. (This implies, as Abel knew, that each lies in k.)
b. Prove that $k(\beta)/k$ is a quadratic extension. Prove the Claim. *Hint: We've already classified the quadratic extensions of k inside L.*

Exercise 4. Prove that the ring of quaternions

$$\mathbb{H} = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}; i^2 = j^2 = k^2 = ijk = -1\}$$

is a division algebra. In other words, prove that non-zero quaternions are invertible.