# PROBLEMS FROM THE HISTORY OF MATHEMATICS – MIDTERM –

DUE FRIDAY, 3/23/2018

**Exercise 1.** The Erdős–Strauss Conjecture claims that every integer of the form $4/n$ (with $n \geq 3$) has a three-term Egyptian fraction decomposition.

a. What is the minimal $n$ for which the greedy algorithm for $4/n$ gives a decomposition of 4 or more terms?
b. Find a three-term Egyptian fraction for the number $4/n$ from (a).

**Exercise 2.** One of the models of hyperbolic geometry is *Poincaré upper half-plane* $\mathbb{H} := \{(x, y) \in \mathbb{R}^2 : y > 0\}$. In this space, straight lines are vertical lines and semicircles that meet the line $y = 0$ at right angles. The area of a region $R$ is defined as the integral

$$\iint_R 1 \frac{dxdy}{y^2}.$$

Find the area of the *ideal hyperbolic triangle* with vertices at 0, 1, and $\infty$.

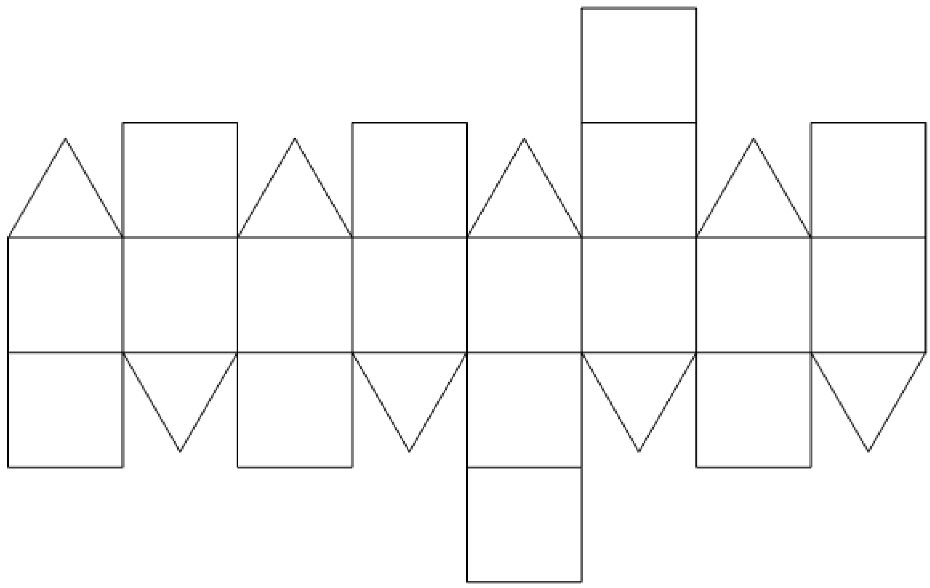
**Exercise 3.** *Fermat's Christmas Theorem* (first proven by Euler c. 1752-1755) states that every prime congruent to 1 mod 4 may be written as the sum of two squares.

a. Prove that no number congruent to 3 mod 4 may be written as the sum of two squares.
b. Prove that no number congruent to 7 mod 8 may be written as a sum of three squares.
c. Suppose that $n$ is perfect. Assuming Fermat's Christmas Theorem, prove that $n$ is a sum of two squares if and only if it is odd.

**Exercise 4.** The angle $\frac{\pi}{3}$ is not trisectable using ruler and compass because $\cos(\frac{\pi}{9})$ is a root of an irreducible cubic.

a. Find the minimal polynomial $f(x) \in \mathbb{Z}[x]$ of $\cos(\frac{\pi}{9})$.
b. Identify the three roots of $f(x)$ in terms of trigonometric constants.
c. Use the cubic formula to find a root of $f(x)$ in radicals.

**Exercise 5.** The *elongated square gyrobicupola*, also known as the *pseudorhombicuboctahedron*, is the solid formed by folding up the following diagram into a closed figure:



This solid consists of regular polygons and the vertex configuration is identical at each vertex. Despite this, it is **not** an Archimedean solid by our definition. Explain which property of Archimedean solids fails.[1]

**Exercise 6.** State and prove theorems about unique factorization in the polynomial rings $\mathbb{R}[x]$ and $\mathbb{C}[x]$.

**Exercise 7.** The $n$th cyclotomic polynomial $\Phi_n(x)$ is defined as the monic polynomial which has as roots the primitive $n$th roots of unity.

a. Prove that
$$\prod_{d|n} \Phi_d(x) = x^n - 1.$$
b. Prove that $\Phi_n(x)$ has constant coefficient 1 except when $n = 1$.
c. Use (a) and (b) to prove that $\Phi_n(x) \in \mathbb{Z}[x]$. *Hint: Induct on $n$.*

**Exercise 8.** Tunnell's theorem has a second case which applies to even, square-free integers. Namely, suppose that $n$ is even, square-free, and congruent. Then the sets
$$C_n = \{(x, y, z) \in \mathbb{Z}^3 : n = 8x^2 + 2y^2 + 64z^2\}$$
$$D_n = \{(x, y, z) \in \mathbb{Z}^3 : n = 8x^2 + 2y^2 + 16z^2\}$$

[1]This solid was repeatedly discovered by geometers incorrectly assembling models of the rhombicuboctahedron.

satisfy $2\#C_n = \#D_n$. (Moreover, the converse holds under the Birch–Swinnerton-Dyer conjecture.)

a. Use Tunnell's theorem to prove that 2 is not a congruent number.
b. Prove that 2 is not a congruent number without appealing to Tunnell's theorem. *Hint: Use a theorem about Pythagorean triples with sides which are squares or twice squares.*

**Exercise 9.**

a. Use an angle duplication formula to prove that

$$\sin x = 2^n \sin\left(\frac{x}{2^n}\right) \prod_{i=1}^{n} \cos\left(\frac{x}{2^i}\right).$$

b. Use (a) to prove *Viète's formula* (1593):

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2+\sqrt{2}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \cdots$$

**Exercise 10.**

a. Prove that

$$\int_1^x \frac{dt}{t} = \lim_{n\to\infty} n(1 - x^{-\frac{1}{n}})$$

by performing a Riemann sum approximation to the integral using the $n$ rectangles with bases $[1, x^{1/n}]$, $[x^{1/n}, x^{2/n}], \ldots, [x^{(n-1)/n}, x]$. This result was known to Euler.

b. Show that your limit is equivalent to the limit

$$\lim_{z\to 0} \frac{x^z - 1}{z}.$$

Use the limit definition of the derivative and standard results from calculus to prove that this limit is $\log x$.